Final Research Project

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This project is based on *"Stability analysis of prey-predator model incorporating prey refuge"* (2005) by Tapan Kumar Kar (see attached copy of original work).

Dynamical systems of predator-prey interaction are a cornerstone of mathematical biology. Many natural situations studied in ecology involve the interaction between two or more species. This can include species in competition for resources, or a predator-prey interaction such as the one studied in this paper. Also known as the Lotka-Volterra equations, predator-prey systems describe the change in population over time for the two species. An example of a system of competing species was discussed during class in Math 361 using rabbits and sheep. To adapt this to a predator-prey context, rabbits and foxes could be used instead.

The basic predator-prey model was proposed in 1910 by Alfred Lotka and was originally used to describe chemical reactions. It was extended to organic systems by Lotka in 1920, before first being used for actual predator-prey interaction in Lotka's book *Biomathematics* (1925). Since its inception, the model has been extended in myriad ways to account for additional components and characteristics of natural systems, with the overall goal of increasing the validity of the model and the degree to which it represents reality. The model has also been used in other disciplines such as economics.

One extension of the model to improve applicability to certain ecological systems was studied by Tapan Kumar Kar. As a mathematics student particularly interested in applications within both ecology and epidemiology, I chose this paper due to the crucial role that predator-prey interactions play in this area of applied mathematics. In this paper, an additional term was added to the basic predator-prey model to account for a spatial refuge available to a certain proportion of the prey population. The parameter *m* represents this proportion. It is assumed that this refuge completely protects prey from harm and is inaccessible to predators. This then reduces the number of prey available to the predators to the value of (1-m)*x, where x is the entire prey population. Previous research finds that prey refuge has a stabilizing effect on predator-prey interaction, but this has not been proved to occur in all cases. As stated by Kar, previous research has also found that a model incorporating refuge of constant proportion does not alter stability, whereas a refuge of constant number does.

Prey refuge can be represented by a variety of different natural situations. For example, it could be a region too small for larger prey to enter, such as a cave or hole, allowing the prey to escape to safety. Additionally, it could be a region with a lower population of predators due to resource availability or hunting policy, which would then act as a refuge area for prey. In all cases, incorporating prey refuge affects the equilibrium, stability, and the overall qualitative behaviour of the system.

The following system of differential equations is used for the model:

$$\frac{dx}{dt} = \alpha x\left(1 - \frac{x}{k}\right) - \frac{\beta(1-m)yx}{1+a(1-m)x},$$
$$\frac{dy}{dt} = -\gamma y + \frac{c\beta(1-m)xy}{1+a(1-m)x}.$$

As we can see from the model, the *x* variable represents the prey population because the interaction term is negative. Conversely, the interaction term is positive for *y* variable, which represents the predator population. All parameters, with the exception of the *m* parameter, are the same as the base version of this predator-prey interaction. Therefore, *α* represents the intrinsic growth rate and *k* represents the carrying capacity for the prey. The value of α would be calculated based upon the reproductive characteristics for the species of study, and the *k* value would represent the maximum possible population, which is similar to its usage in the logistic equation. *γ* is the death rate for the predator population. *β/a* is the maximum amount of prey that can be eaten by the predators per unit time. Finally, *c* is the conversion factor representing how many new predators would be born for each captured prey.

The system parameter is the refuge parameter *m*. Varying this parameter represents changing the proportion of prey protected by the spatial refuge and observing the resulting effect on the overall populations. The original work presented numerical simulations, which have been reproduced for this project using XPPAUTO. Solution curves and phase portrait diagrams are presented for various values of *m*, which represent the dynamics of the system based on the proportion of prey that is able to use the refuge. A nullcline diagram and a bifurcation diagram for parameter *m* are also presented.

For the numerical simulations, identical parameter values were used in XPPAUTO as were used in the original paper. Therefore, let $\alpha=10$, $k=100$, $a=0.02$, $\gamma=0.09$, $\beta=0.6$, and $c=0.02$ for all simulations presented. Parameter *m* was varied between 0 and 1 as the system parameter. In the first diagram (Figure 1), x and y nullclines are shown for the system as well as the interior equilibrium point (9.80, 19.65). Analysis was focused on this equilibrium point in the original work as this is the only steady state in the system in which both predator and prey populations are nonzero, and thus is of the most interest when studying the effects of the refuge.

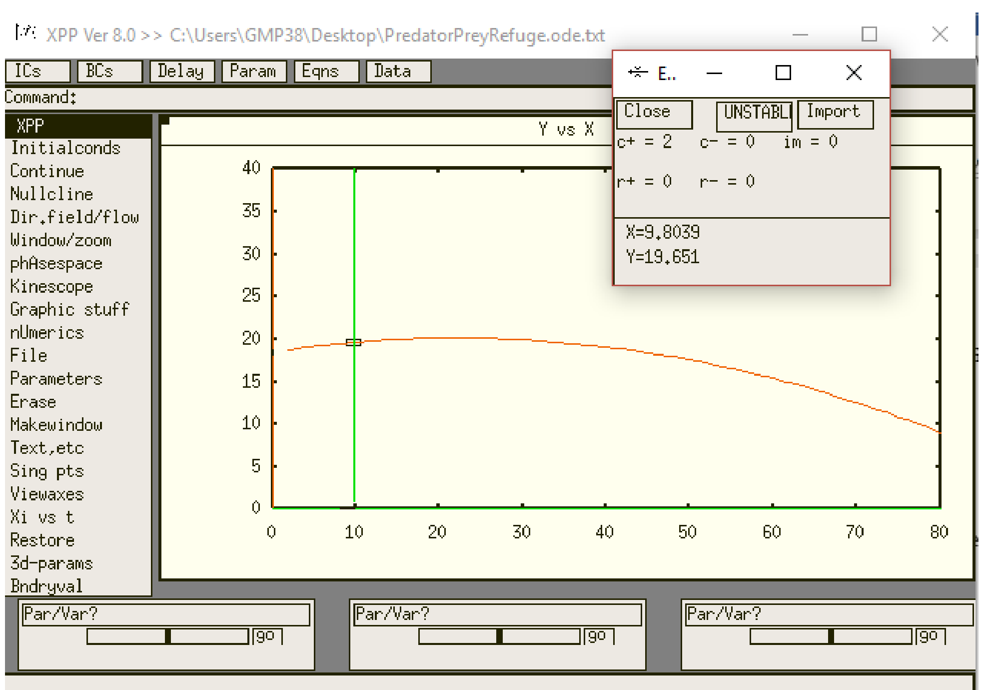

Figure 1: Nullclines with Equilibrium Point shown (9.80, 19.651). m=0.1.

Figure 2 shows the phase portrait for m=0.1. This corresponds to 10% of the prey population accessing the spatial refuge. From the diagram it can be seen that the interior equilibrium point (9.80, 19.65) is unstable and solutions approach a unit limit cycle. Solutions outside the limit cycle spiral inward, while solutions inside the limit cycle spiral outward, before both converge onto the limit cycle over time.

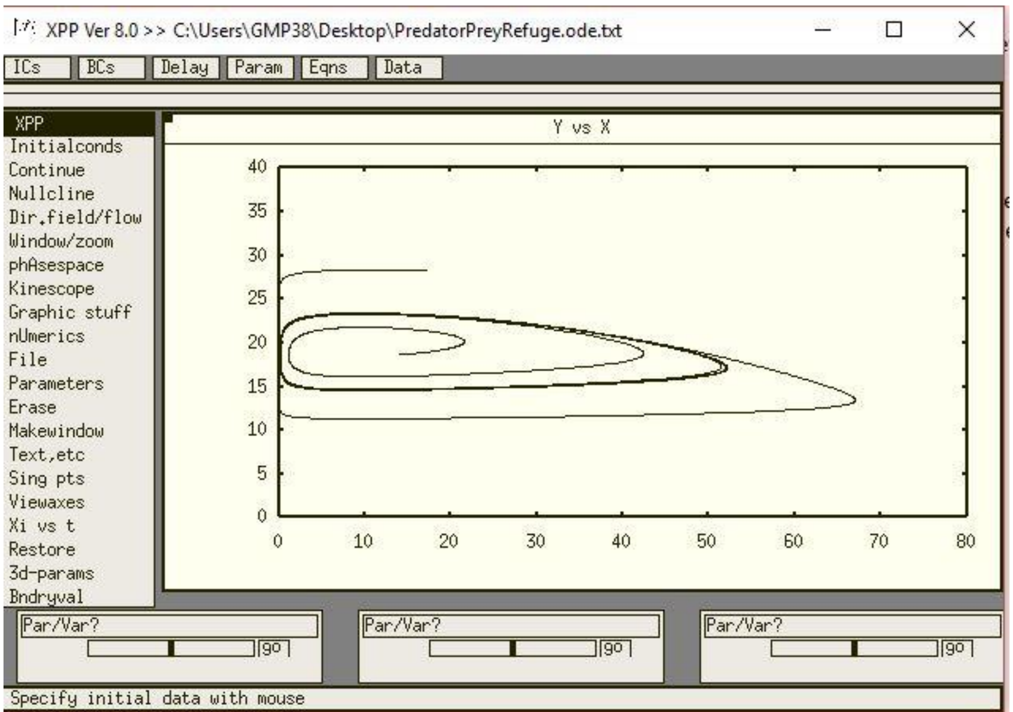

Figure 2: Phase Portrait of system with m=0.1

Similar to Figure 1, Figure 3 also shows an unstable interior equilibrium point and shows solutions of the system trending towards a unique limit cycle. In this case, the refuge parameter m=0.3.

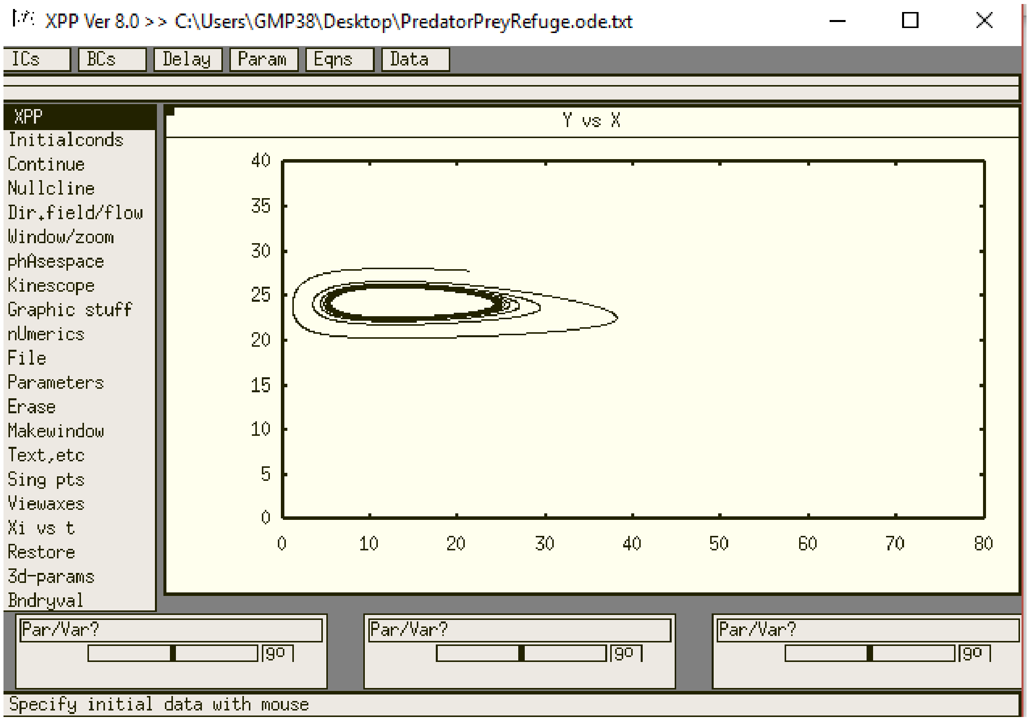

Figure 3: Phase Portrait of system with m=0.3

Figure 4 shows the solution curves over time. In this diagram both the x and y variables (predator and prey populations) are plotted against time. The value m=0.32 is used, which is the bifurcation point for the system. We see that in this case, both predator (black line) and prey (red line) populations display periodic solutions.

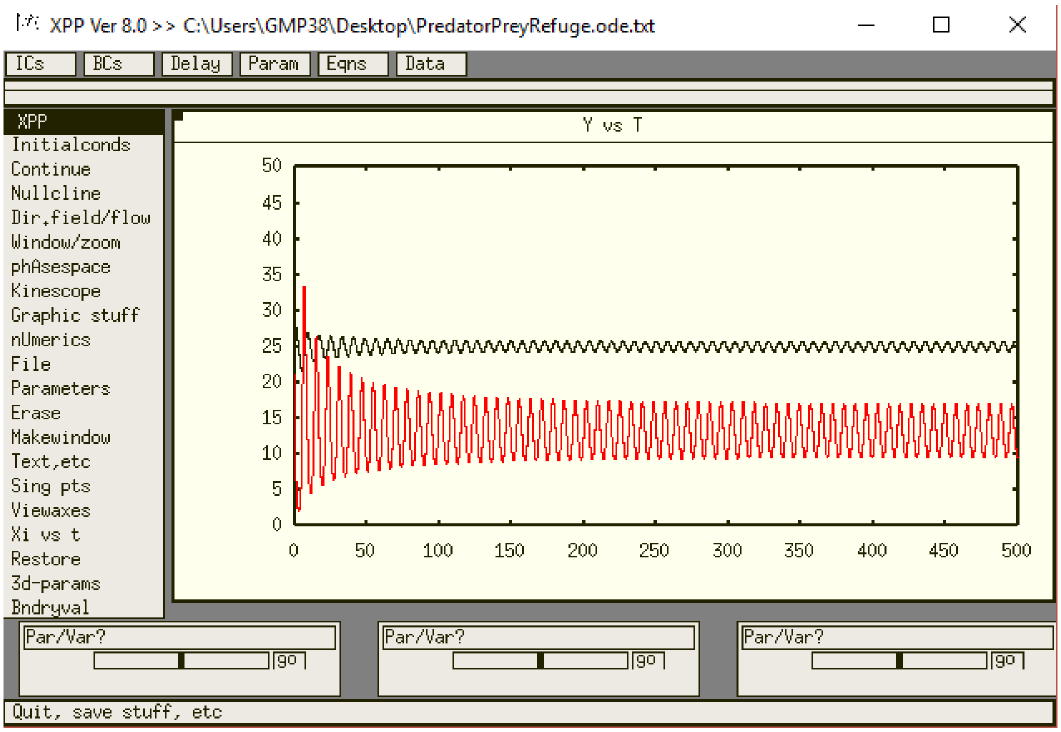

Figure 4: Solution curves over time for m=0.32 (at the bifurcation point). Predator: Black Line; Prey: Red Line

Once the value of *m* is increased past the bifurcation point, the periodic solutions disappear and the system trends toward a stable equilibrium, solutions curves for which are seen in Figure 5 (m=0.4) and Figure 7 (m=0.85). Additionally, Figure 6 shows the phase portrait for the system with the predator population plotted against the prey population. From this, we can see that all solutions spiral toward the attractor point (17.65, 32.3), which implies that the system is stable over time at this parameter value.

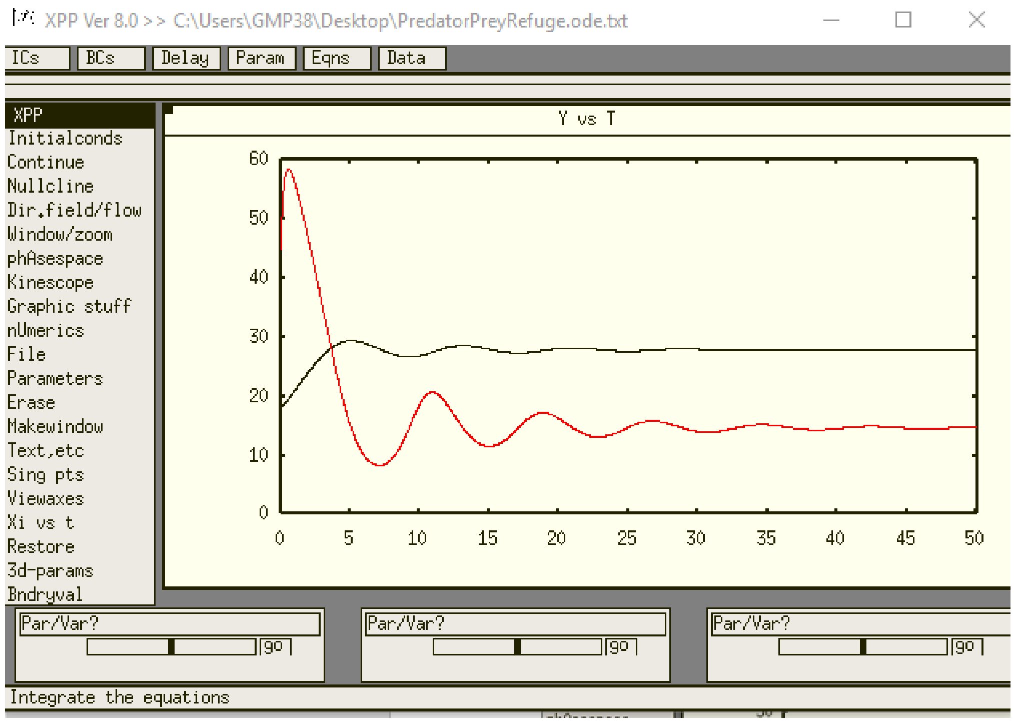

Figure 5: Solution curves over time for m=0.4. Predator: Black Line; Prey: Red Line

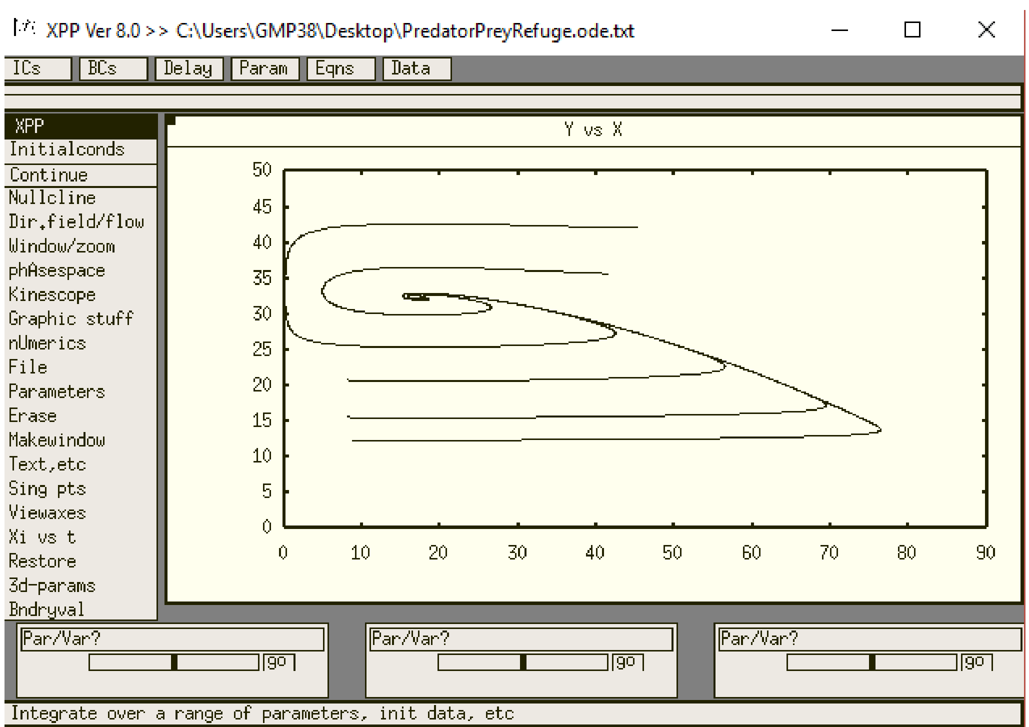

Figure 6: Phase Portrait for m=0.5. All solutions tend to (17.65, 32.3).

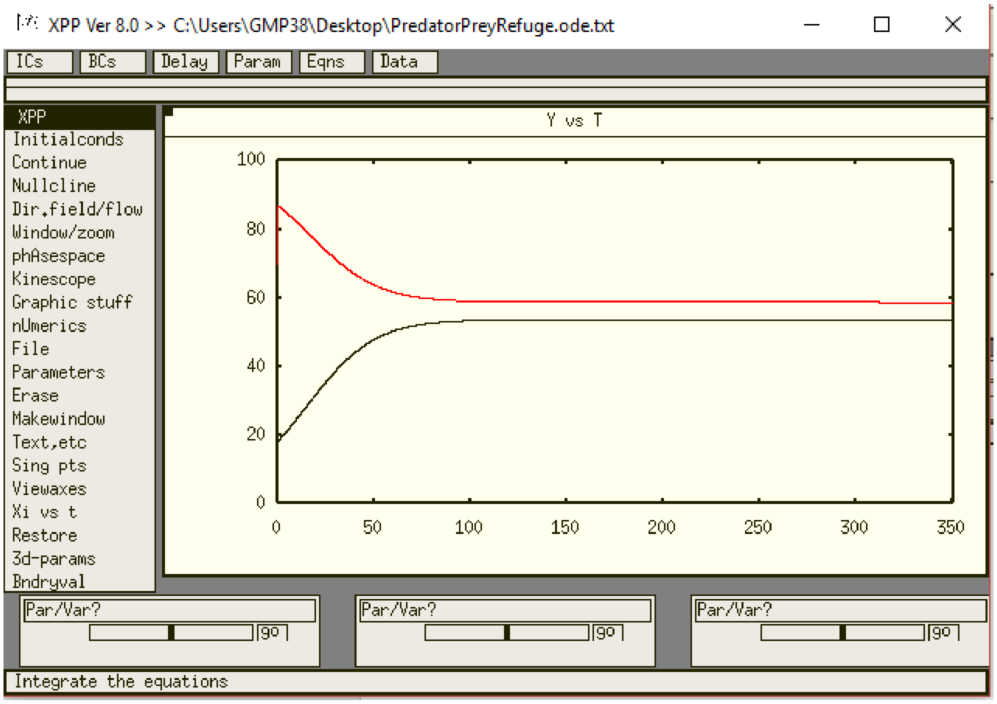

Figure 7: Solution curves over time for m=0.85. Predator: Black Line-tends to 53.82; Prey: Red Line-tends to 58.82.

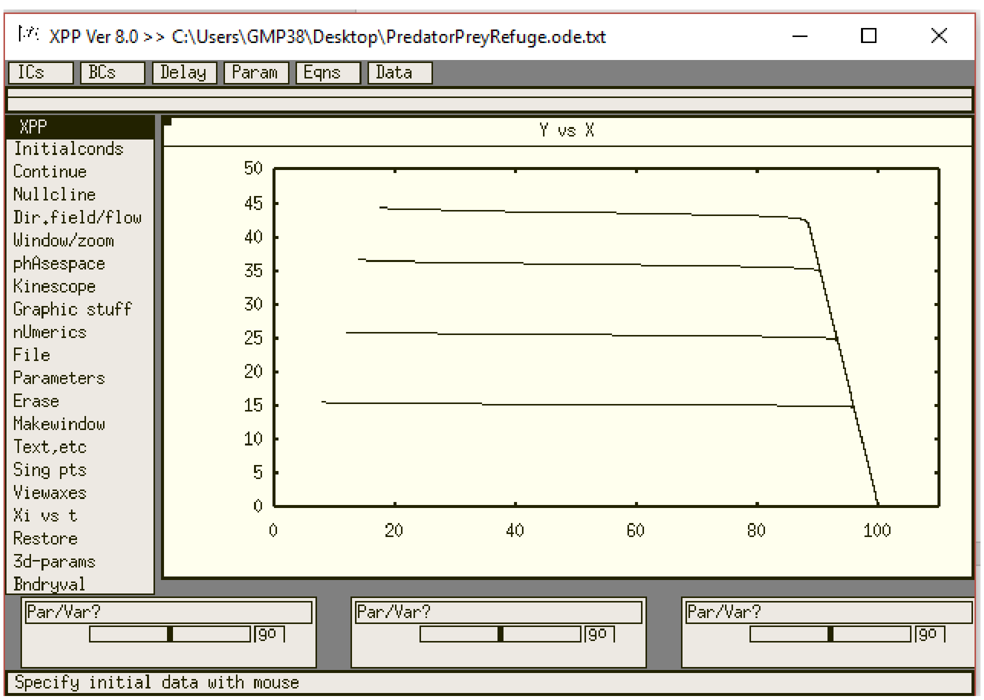

Figure 8: Phase Portrait of the system with m=0.95. All solutions tend to (100,0).

As seen in Figure 8, if the bifurcation parameter is increased further to m=0.95, the system still has an equilibrium point, but it is no longer in the interior of the system. Instead, all solutions tend to the attractor point (100, 0). This represents that if 95% of the prey population has access to the refuge, the predator population will not be able to sustain itself and will eventually die out, regardless of the initial population values. Intuitively, this makes sense as such a large proportion of the prey being inaccessible to predators would result in a critical shortage of food for predators.

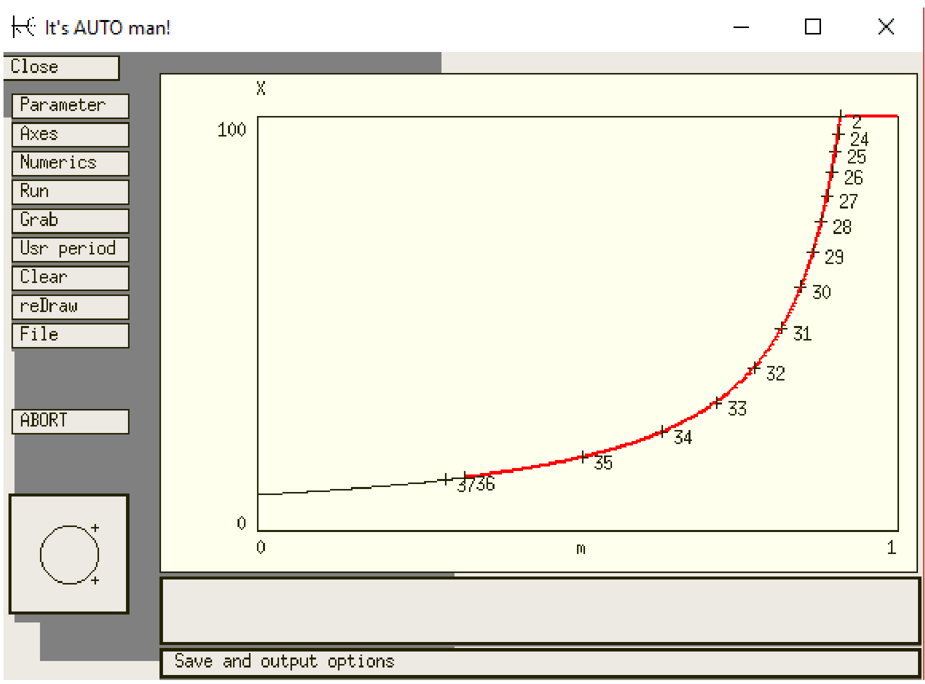

Figure 9: Bifurcation diagram for prey population with parameter m.

Finally, Figures 9 & 10 display bifurcation diagrams for parameter m. Figure 9 represents the prey population and Figure 10 represents the predator population. We can see from the diagram that the prey population strictly increases as the value of *m* increases and increases at a very rapid rate once m is beyond the value of β, which for this simulation was taken as β=0.6.

This is consistent with findings in the original paper that the system is stable for m>β and tends toward the boundary equilibrium point. In the real-life context of the model, this also makes sense as a greater number of prey able to use the refuge would logically result in a greater overall prey population remaining alive.

Figure 10 shows that the predator population increases up to a point while *m* increases, before dropping off rapidly to 0 once too many prey are accessing the spatial refuge so that the predators can no longer sustain themselves. We can also see that in both bifurcation diagrams, the stability changes at the bifurcation point (m=0.32) which is also consistent with previous diagrams as well as the results of the original research.

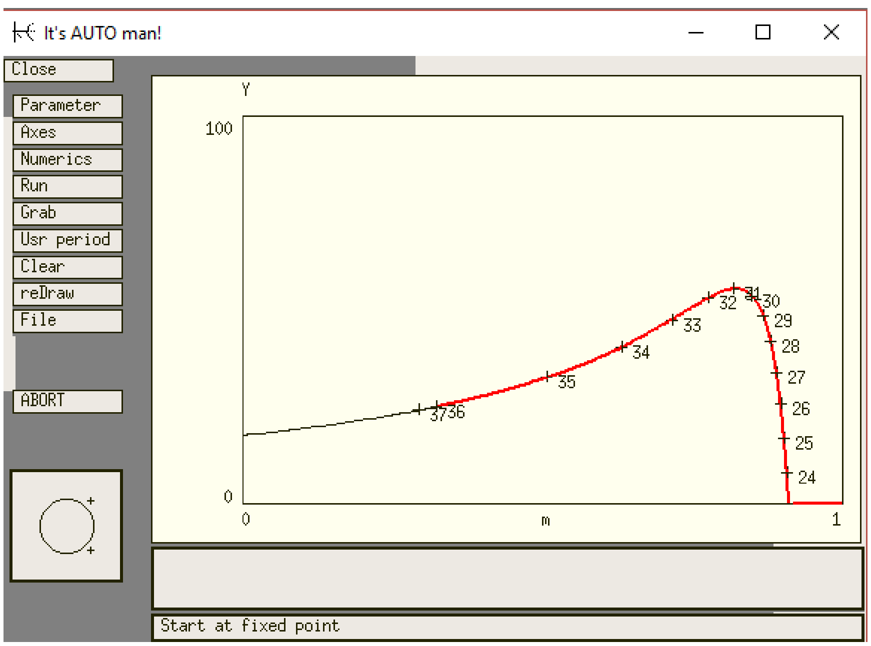

Figure 10: Bifurcation diagram for predator population with parameter m.

Overall, this research was strong and presented new insight into predator-prey interaction by adding a component to the model that has useful applications to many real-life ecological systems. The researcher proved the existence and conditions necessary to produce equilibrium points in the system and determine their stability. The researcher also presented phase portraits and solution curves for values of the system parameter that were less than, equal to, and greater than the bifurcation point. This allowed for a broad picture of the system to be understood, and particularly how the qualitative behaviour changed. The model and diagrams were constructed in a way that allowed for a clear interpretation to be made for what varying the parameter would represent and what it would affect in a real-life context, which I believe strengthens the usefulness of the research. While the paper did include the necessary proofs to maintain rigour, emphasis was placed on numerical simulations. Methods of calculation and diagram construction were consistent with those that were covered in Math 361.

To extend this research, I would incorporate a tier structure to the spatial refuge. Instead of making the assumption that the refuge provided guaranteed safety to prey, the refuge could include multiple tiers, each of which had a progressively lower likelihood of the prey surviving. This may better represent some natural situations as refuges could exist in which prey are more likely to escape predators, but are still not guaranteed safety. For example, this could be an enclosed space which poses physical difficulties for predators to access, such as entrance size or needing to access the space above ground level. However, while prey would certainly be safer in these instances, some such refuges would not be impossible for prey to access, and therefore a certain percentage of prey may still be vulnerable. In mathematical terms, this would involve adding an additional term to what was already added in the previously discussed research. Instead of adding just the (1-m) term to represent the refuge, it could be replaced with (1-m-n/2) or (1-m-n/2-h/4), which would represent two and three tiered refuges respectively. In this example, the *m* parameter would function identically as it did before and would represent the proportion of prey guaranteed safety by the refuge. However, *n* would represent prey in a tier of the refuge that only provided 50% likelihood of safety, and therefore half of these prey would still be vulnerable to predators. Similarly, in the three-tiered example, *h* would represent the proportion of prey for which the refuge was only providing a 25% likelihood of safety.

The following are preliminary diagrams for a two-tiered refuge system with the (1-m-n/2) term replacing the (1-m) term in the model. The new diagrams were also computed using XPPAUTO, using identical parameter values in order to determine the effect of the two-tier refuge. Figure 11 shows the new nullcline diagram and phase portrait and equilibrium point (10.381, 20.673) for parameter values m=0.1 and n=0.1. We can see that the nullclines and the behaviour of the system are similar to the original system, with the equilibrium point having moved from (9.80, 19.65) to (10.381, 20.673). In figures 12, 13, and 14 we see solution curves plotted for predator (black line) and prey (red line) populations for different parameter values for *m* and *n.* For figure 12, m=0.2 and n=0.2 and periodic solutions are observed. However, in figure 13, m=0.25 and n=0.2 and the populations tend toward equilibrium values. In figure 14, the parameter values are switched so m=0.2 and n=0.25, and the system appears to again tend toward an equilibrium, but at a much slower rate. This suggests that parameter *m* has a larger impact on the behaviour of the system. This also suggests that each new tier added to the refuge that provided less likelihood of safety would have decreasing impact on the overall behaviour of the system.

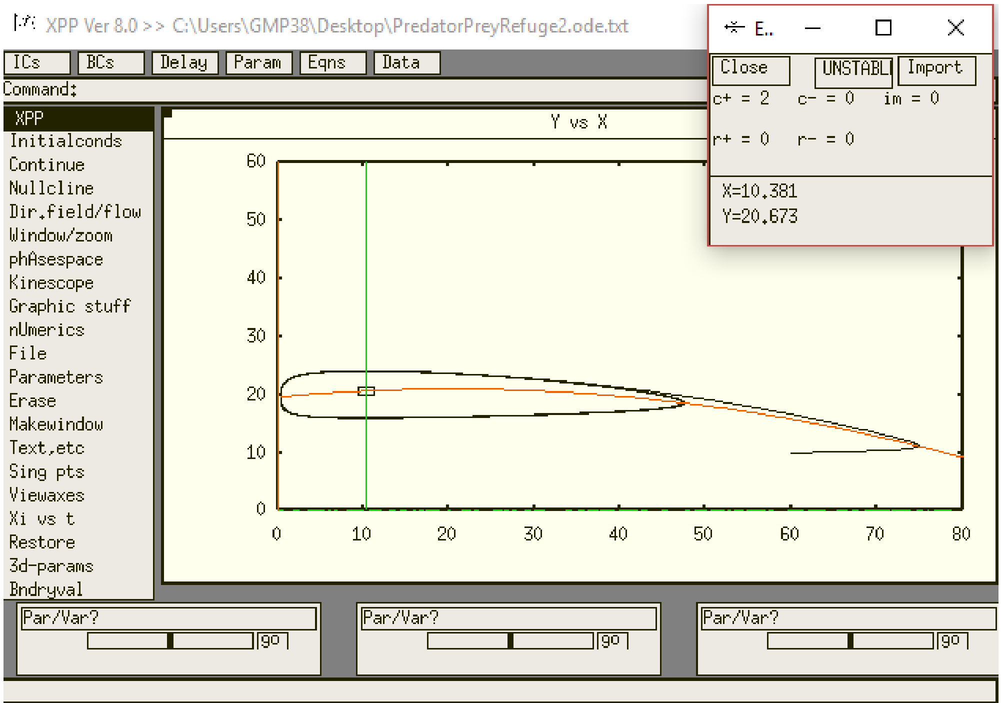

Figure 11: Nullcline diagram & Phase Portrait for two-tiered refuge with parameters m=0.1 and n=0.1. The new (unstable) equilibrium point (10.381, 20.673) is shown.

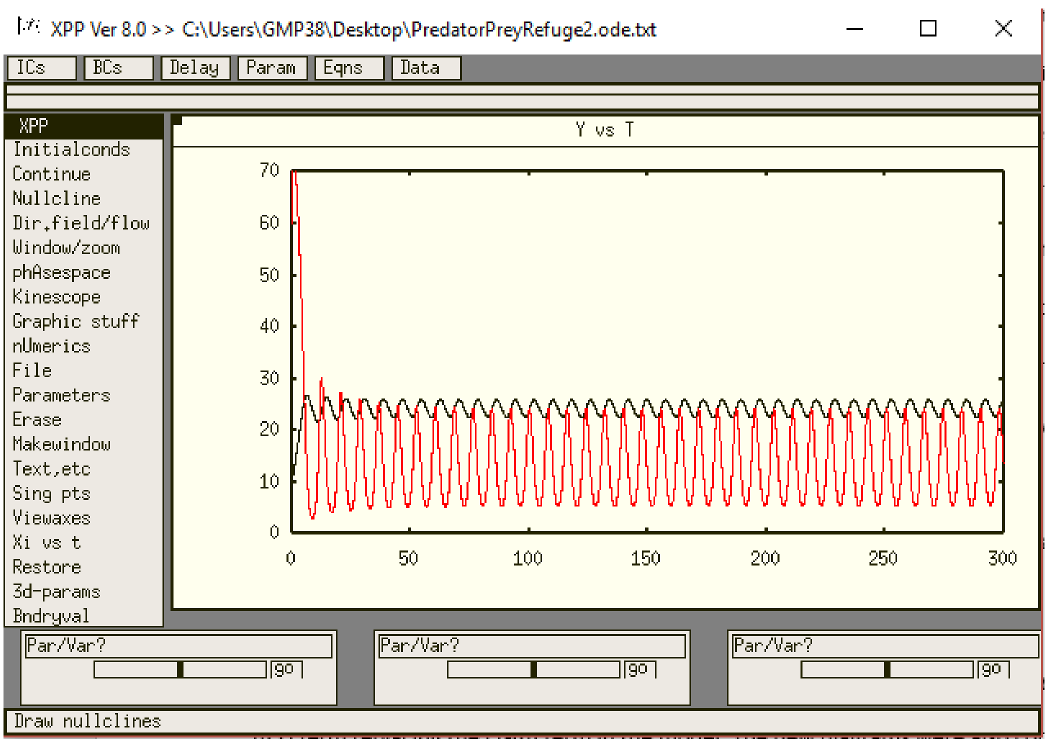

Figure 12: Solution curves for m=0.2 and n=0.2 in two-tiered refuge system

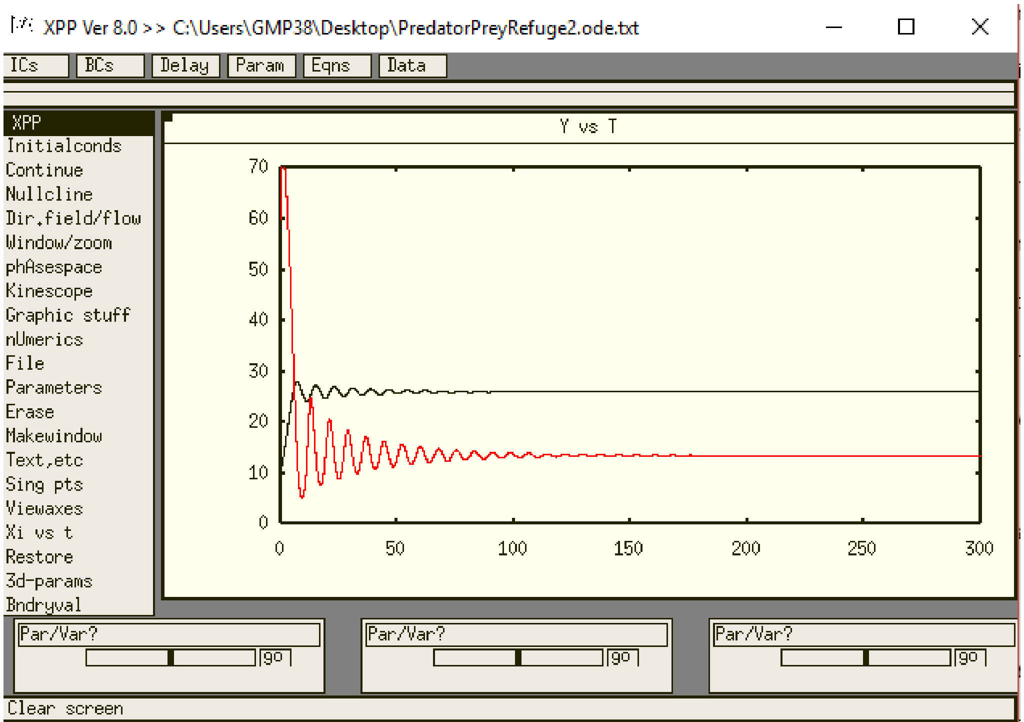

Figure 13: Solution curves for m=0.25 and n=0.2 in two-tiered refuge system

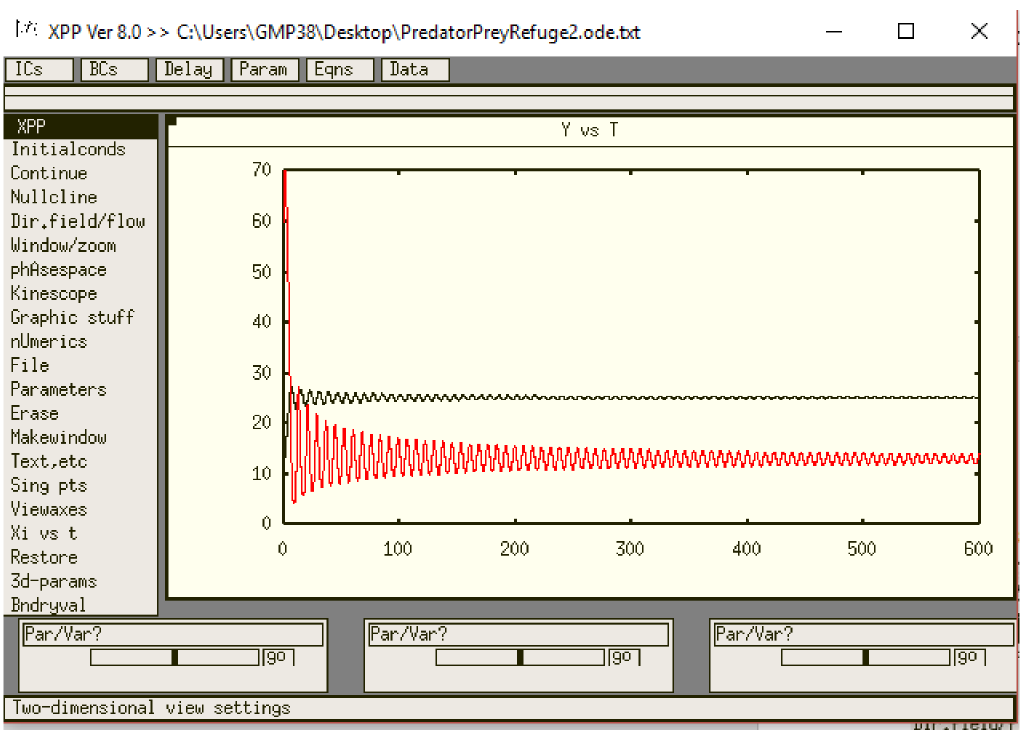

Figure 14: Solution curves for m=0.2 and n=0.25 in two-tiered refuge system